Early Journal Content on JSTOR, Free to Anyone in the World

This article is one of nearly 500,000 scholarly works digitized and made freely available to everyone in the world by JSTOR.

Known as the Early Journal Content, this set of works include research articles, news, letters, and other writings published in more than 200 of the oldest leading academic journals. The works date from the mid-seventeenth to the early twentieth centuries.

We encourage people to read and share the Early Journal Content openly and to tell others that this resource exists. People may post this content online or redistribute in any way for non-commercial purposes.

Read more about Early Journal Content at http://about.jstor.org/participate-jstor/individuals/early-journal-content.

JSTOR is a digital library of academic journals, books, and primary source objects. JSTOR helps people discover, use, and build upon a wide range of content through a powerful research and teaching platform, and preserves this content for future generations. JSTOR is part of ITHAKA, a not-for-profit organization that also includes Ithaka S+R and Portico. For more information about JSTOR, please contact support@jstor.org.

On the Syzygies of the Binary Sextic and their Relations.

By J. Hammond.

(I). *The Irreducible Syzygies of the Binary Sextic, as far as the eighth degree, with the Linear Relations connecting Compound Syzygies, as far as the ninth degree.*

1. Following Prof. Cayley's notation, "Tables for the Binary Sextic" (*American Journal of Mathematics*, Vol. IV, p. 380), the capital letters A, B, ... Z are used to denote the 26 covariants. These are identical with those given by Prof. Cayley, with two exceptions, viz. J' and R', which are connected with them by the relations $J' = BC - J$, $8R' = BK - 6Q + R$.

The syzygies are as follows:

Deg. Order.
(5.16) $AL + CH - DG = 0$
(6.8) $AM - 2ABE + B^2D - BC^2 - 3CJ' - 36DI + 36EF = 0$
(6.10) $AN - CK + EG = 0$
(6.12) $3A^2I - 3ABF + 2ACE + BCD - C^3 - 3DJ' + 27F^2 = 0$
(6.14) $AO - 2CL + 3FG = 0$
(6.14) $AO - DK + EH = 0$
(6.16) $A^2J' - A^2BC + 12ACF + 4C^2D + G^2 = 0$
(6.18) $A^2K - ABH + ACG - 12DL + 18FH = 0$
(6.20) $A^3E - A^2BD - 2A^2C^2 + 18ADF + 12CD^2 + 3GH = 0$
(6.24) $A^3F - A^2CD + 4D^3 + H^2 = 0$
(7.6) $9AP + 12ABI - 3B^2F + 4BCE - CM - 12EJ' + 108FI = 0$
(7.10) $12ACI + AE^2 - 3BCF + 4C^2E - DM - 9FJ' = 0$
(7.12) $AQ + 2EL - 3FK = 0$
(7.12) $AR' - GJ' - 2CO = 0$
(7.12) $CO - DN - EL = 0$
(7.12) $ABK - B^2H + BCG - 6CO - 12EL - 18FK + 36HI = 0$
(7.14) $A^2M - A^2BE - 6ACJ' + 18AEF + 3GK + 12CDE = 0$
(7.16) $ACK - BCH + C^2G - 6DO - 18FL = 0$
(7.16) $2ACK + AEG - ABL - 6DO - 3HJ' = 0$
(7.18) $A^2CE + 2ABCD - 2AC^3 - 3ADJ' - 18CDF - 3GL = 0$

Deg. Order.

(7.18) $2A^3I - A^2BF + A^2CE - 2ADJ' + 4D^2E + HK = 0$

(7.22) $A^2DE - 3A^2CF - ABD^2 + AC^2D + 18D^2F + 3HL = 0$

(8.8) $AS + 4C^2I - FM - J'^2 = 0$

(8.8) $4AEI - 4BDI + 4C^2I - 2BEF + 3CE^2 - FM - 3DP = 0$

(8.10) $AT - 2EO - J'K = 0$

(8.10) $CQ + EO - 3FN = 0$

(8.10) $3BCK - BEG - 6CR' + 6EO - 36FN + GM - 3J'K = 0$

(8.10) $2B^2L - BCK - 3BEG - 18EO + GM - 72IL + 3J'K = 0$

(8.12) $3A^2P + 4A^2BI - 4AEJ' + 4DE^2 + K^2 = 0$

(8.12) $2ABCE + ACM - 3AEJ' - 6C^2J' - 18CEF + 3GN = 0$

(8.14) $ABO - ACN + 3AGI - 9FO - 3J'L = 0$

(8.14) $AEK - BEH + CEG + 6DQ - 18FO = 0$

(8.14) $BCL - 2C^2K - CEG + 3DR' - 3J'L = 0$

(8.14) $ABO + 2ACN - AEK + 2BCL - 4C^2K - 18FO + HM = 0$

(8.16) $GO - HN - KL = 0$

(8.16) $4A^2CI - ABCF + 2AC^2E - 3AFJ' - 2CDJ' + GO = 0$

(8.16) $6A^2CI - 3ABCF + ABDE + 3AC^2E - ADM - 18DEF - 3KL = 0$

(8.20) $A^2DI - ABDF + AC^2F + 9DF^2 + L^2 = 0$

(8.20) $AGK - BGH + CG^2 - 6HO - 12L^2 = 0.$

2. By a Compound Syzygy is meant either a simply divisible syzygy, *i. e.* one which is divisible by a power or product of the covariants A, B, . . . Z, or else a linear function of simply divisible syzygies. The consideration of the simply divisible syzygies will serve the purpose of the present article.

The linear relations connecting them are as follows:

Deg. Order.

(8.18)
$$\begin{aligned}&A(CO - DN - EL) - C(AO - DK + EH) + D(AN - CK + EG)\\&+ E(AL + CH - DG) = 0\end{aligned}$$

(8.22)
$$\begin{aligned}&A(ACK - BCH + C^2G - 6DO - 18FL) - C(A^2K - ABH + ACG - 12DL + 18FH)\\&+ 6D(AO - 2CL + 3FG) + 18F(AL + CH - DG) = 0\end{aligned}$$

(8.24)
$$\begin{aligned}&A(A^2CE + 2ABCD - 2AC^3 - 3ADJ' - 18CDF - 3GL)\\&- C(A^3E - A^2BD - 2A^2C^2 + 18ADF + 12CD^2 + 3GH)\\&+ 3D(A^2J' - A^2BC + 12ACF + 4C^2D + G^2) + 3G(AL + CH - DG) = 0\end{aligned}$$

(8.28)
$$\begin{aligned}&A(A^2DE - 3A^2CF - ABD^2 + AC^2D + 18D^2F + 3HL) + 3C(A^3F - A^2CD + 4D^3 + H^2)\\&- D(A^3E - A^2BD - 2A^2C^2 + 18ADF + 12CD^2 + 3GH) - 3H(AL + CH - DG) = 0\end{aligned}$$

(9.14)
$$\begin{aligned}&3A(4AEI - 4BDI + 4C^2I - 2BEF + 3CE^2 - FM - 3DP)\\&- C(12ACI + AE^2 - 3BCF + 4C^2E - DM - 9FJ')\\&+ D(9AP + 12ABI - 3B^2F + 4BCE - CM - 12EJ' + 108FI)\\&- 4E(3A^2I - 3ABF + 2ACE + BCD - C^3 - 3DJ' + 27F^2)\\&+ 3F(AM - 2ABE + B^2D - BC^2 - 3CJ' - 36DI + 36EF) = 0\end{aligned}$$

Deg. Order.

(9.16)
$$\begin{aligned}&A(CQ+EO-3FN)-C(AQ+2EL-3FK)-E(AO-2CL+3FG)\\&\quad+3F(AN-CK+EG)=0\end{aligned}$$

(9.16)
$$\begin{aligned}&A(3BCK-BEG-6CR'+6EO-36FN+GM-3J'K)\\&\quad+A(2B^2L-BCK-3BEG-18EO+GM-72IL+3J'K)\\&\quad-2B^2(AL+CH-DG)+6C(AR'-GJ'-2CO)\\&\quad-2C(ABK-B^2H+BCG-6CO-12EL-18FK+36HI)\\&\quad+12E(AO-2CL+3FG)+36F(AN-CK+EG)\\&\quad-2G(AM-2ABE+B^2D-BC^2-3CJ'-36DI+36EF)\\&\quad+72I(AL+CH-DG)=0\end{aligned}$$

(9.18)
$$\begin{aligned}&A(2ABCE+ACM-3AEJ'-6C^2J'-18CEF+3GN)\\&\quad-C(A^2M-A^2BE-6ACJ'+18AEF+12CDE+3GK)\\&\quad+3E(A^2J'-A^2BC+12ACF+4C^2D+G^2)-3G(AN-CK+EG)=0\end{aligned}$$

(9.20)
$$\begin{aligned}&A(AEK-BEH+CEG+6DQ-18FO)-6D(AQ+2EL-3FK)\\&\quad-E(A^2K-ABH+ACG-12DL+18FH)+18F(AO-DK+EH)=0\end{aligned}$$

(9.20)
$$\begin{aligned}&A(BCL-2C^2K-CEG+3DR'-3J'L)+C(2ACK+AEG-ABL-6DO-3HJ')\\&\quad-3D(AR'-GJ'-2CO)+3J'(AL+CH-DG)=0\end{aligned}$$

(9.20)
$$\begin{aligned}&A(ABO-ACN+3AGI-9FO-3J'L)-AB(AO-2CL+3FG)\\&\quad+AC(AN-CK+EG)-BC(AL+CH-DG)\\&\quad-C(ACK-BCH+C^2G-6DO-18FL)\\&\quad+C(2ACK+AEG-ABL-6DO-3HJ')+9F(AO-2CL+3FG)\\&\quad-G(3A^2I-3ABF+2ACE+BCD-C^3-3DJ'+27F^2)\\&\quad+3J'(AL+CH-DG)=0\end{aligned}$$

(9.20)
$$\begin{aligned}&A(ABO+2ACN-AEK+2BCL-4C^2K-18FO+HM)-AB(AO-DK+EH)\\&\quad-2AC(AN-CK+EG)-BC(AL+CH-DG)\\&\quad+C(2ACK+AEG-ABL-6DO-3HJ')\\&\quad-D(ABK-B^2H+BCG-6CO-12EL-18FK+36HI)\\&\quad+18F(AO-DK+EH)+E(A^2K-ABH+ACG-12DL+18FH)\\&\quad-H(AM-2ABE+B^2D-BC^2-3CJ'-36DI+36EF)=0\end{aligned}$$

(9.22)
$$\begin{aligned}&A(GO-HN-KL)-G(AO-DK+EH)+H(AN-CK+EG)\\&\quad+K(AL+CH-DG)=0\end{aligned}$$

(9.22)
$$\begin{aligned}&A(6A^2CI-3ABCF+ABDE+3AC^2E-ADM-18DEF-3KL)\\&\quad-3C(2A^3I-A^2BF+A^2CE-2ADJ'+4D^2E+HK)\\&\quad+D(A^2M-A^2BE-6ACJ'+18AEF+12CDE+3GK)\\&\quad+3K(AL+CH-DG)=0\end{aligned}$$

(9.22)
$$\begin{aligned}&3A(4A^2CI-ABCF+2AC^2E-3AFJ'-2CDJ'+GO)\\&\quad-A^2(12ACI+AE^2-3BCF+4C^2E-DM-9FJ')\\&\quad-D(A^2M-A^2BE-6ACJ'+18AEF+3GK+12CDE)\\&\quad+E(A^3E-A^2BD-2A^2C^2+18ADF+12CD^2+3GH)-3G(AO-DK+EH)=0\end{aligned}$$

Deg. Order.

$$\begin{aligned}
(9.22)\quad & 3A\,(4A^2CI - ABCF + 2AC^2E - 3AFJ' - 2CDJ' + GO) \\
& - 4AC\,(3A^2I - 3ABF + 2ACE + BCD - C^3 - 3DJ' + 27F^2) \\
& + 2C\,(A^2CE + 2ABCD - 2AC^3 - 3ADJ' - 18CDF - 3GL) \\
& + 9F\,(A^2J' - A^2BC + 12ACF + 4C^2D + G^2) - 3G\,(AO - 2CL + 3GF) = 0
\end{aligned}$$

$$\begin{aligned}
{}^*(9.24)\quad & A^2\,(ABK - B^2H + BCG - 6CO - 12EL - 18FK + 36HI) \\
& - AB\,(A^2K - ABH + ACG - 12DL + 18FH) - 6AC\,(AO - 2CL + 3FG) \\
& + 12AC\,(CO - DK + EH) + 12AE\,(AL + CH - DG) - 12C^2\,(AL + CH - DG) \\
& - 12D\,(ACK - BCH + C^2G - 6DO - 18FL) \\
& + 12D\,(2ACK + AEG - ABL - 6DO - 3HJ') \\
& + 18F\,(A^2K - ABH + ACG - 12DL + 18FH) \\
& - 12H\,(3A^2I - 3ABF + 2ACE + BCD - C^3 - 3DJ' + 27F^2) = 0
\end{aligned}$$

$$\begin{aligned}
(9.26)\quad & 3A\,(A^2DI - ABDF + AC^2F + 9DF^2 + L^2) \\
& - AD\,(3A^2I - 3ABF + 2ACE + BCD - C^3 - 3DJ' + 27F^2) \\
& + C\,(A^2DE - 3A^2CF - ABD^2 + AC^2D + 18D^2F + 3HL) \\
& + D\,(A^2CE + 2ABCD - 2AC^3 - 3ADJ' - 18CDF - 3GL) \\
& - 3L\,(AL + CH - DG) = 0
\end{aligned}$$

$$\begin{aligned}
(9.26)\quad & A\,(AGK - BGH + CG^2 - 6HO - 12L^2) - G\,(A^2K - ABH + ACG - 12DL + 18FH) \\
& + 6H\,(AO - 2CL + 3FG) + 12L\,(AL + CH - DG) = 0
\end{aligned}$$

$$\begin{aligned}
(9.26)\quad & 2AD\,(3A^2I - 3ABF + 2ACE + BCD - C^3 - 3DJ' + 27F^2) \\
& + 2C\,(A^2DE - 3A^2CF - ABD^2 + AC^2D + 18D^2F + 3HL) \\
& - 3D\,(2A^3I - A^2BF + A^2CE - 2ADJ' + 4D^2E + HK) \\
& + 3E\,(A^3F - A^2CD + 4D^3 + H^2) \\
& - 3F\,(A^3E - A^2BD - 2A^2C^2 + 18ADF + 12CD^2 + 3GH) \\
& + 3H\,(AO - 2CL + 3FG) - 3H\,(AO - DK + EH) = 0
\end{aligned}$$

$$\begin{aligned}
(9.26)\quad & A^3\,(AM - 2ABE + B^2D - BC^2 - 3CJ' - 36DI + 36EF) \\
& - A^2\,(A^2M - A^2BE - 6ACJ' + 18AEF + 12CDE + 3GK) \\
& + AB\,(A^3E - A^2BD - 2A^2C^2 + 18ADF + 12CD^2 + 3GH) \\
& - 3AC\,(A^2J' - A^2BC + 12ACF + 4C^2D + G^2) \\
& + 12AD\,(3A^2I - 3ABF + 2ACE + BCD - C^3 - 3DJ' + 27F^2) \\
& - 12D\,(A^2CE + 2ABCD - 2AC^3 - 3ADJ' - 18CDF - 3GL) \\
& - 18F\,(A^3E - A^2BD - 2A^2C^2 + 18ADF + 12CD^2 + 3GH) \\
& + 3G\,(A^2K - ABH + ACG - 12DL + 18FH) = 0
\end{aligned}$$

$$\begin{aligned}
{}^*(9.28)\quad & A^2\,(2ACK + AEG - ABL - 6DO - 3HJ') \\
& - 2AC\,(A^2K - ABH + ACG - 12DL + 18FH) \\
& - G\,(A^3E - A^2BD - 2A^2C^2 + 18ADF + 12CD^2 + 3GH) - 12CD\,(AL + CH - DG) \\
& + A^2B\,(AL + CH - DG) + 6AD\,(AO - 2CL + 3FG) \\
& + 3H\,(A^2J' - A^2BC + 12ACF + 4C^2D + G^2) = 0
\end{aligned}$$

---

* Also relations of deg. order (9.24), (9.28), (9.30), and (9.34) obtained by multiplying the relations of deg. 8 by the sextic A.

Deg. Order.

*(9.30)
$$\begin{aligned}&3A^2(2A^3I - A^2BF + A^2CE - 2ADJ' + 4D^2E + HK)\\ &- 2A^3(3A^2I - 3ABF + 2ACE + BCD - C^3 - 3DJ' + 27F^2)\\ &- 3AB(A^3F - A^2CD + 4D^3 + H^2)\\ &+ AC(A^3E - A^2BD - 2A^2C^2 + 18ADF + 12CD^2 + 3GH)\\ &- 12D(A^2DE - 3A^2CF - ABD^2 + AC^2D + 18D^2F + 3HL)\\ &+ 54F(A^3F - A^2CD + 4D^3 + H^2)\\ &- 3H(A^2K - ABH + ACG - 12DL + 18FH) = 0\end{aligned}$$

(9.32)
$$\begin{aligned}&A^3(AO - 2CL + 3FG) - A^3(AO - DK + EH)\\ &+ 2A^2C(AL + CH - DG) - AD(A^2K - ABH + ACG - 12DL + 18FH)\\ &- 12D^2(AL + CH - DG) - 3G(A^3F - A^2CD + 4D^3 + H^2)\\ &+ H(A^3E - A^2BD - 2A^2C^2 + 18ADF + 12CD^2 + 3GH) = 0\end{aligned}$$

These relations take the place of Prof. Sylvester's syzygants of the second grade, defined as rational integral functions of irreducible ones of the first grade, which vanish when expressed in terms of the covariants (see *American Journal of Mathematics,* Vol. IV, p. 61). Those of the present article are *linear* functions of the syzygants of the first grade, given in the preceding article, the coefficients being covariants.† They are written at full length to exhibit clearly their property of vanishing identically when expressed in terms of the covariants.‡

In the following articles it will be shown that the list is complete.

(II). *Digression on the Generating Function for Syzygants of Binary Quantics.*

3. If, for the quantic of order i, and for its covariants of deg. order (m, n), there be

α asyzygetic covariants,
β syzygies,
γ groundforms,
δ compound covariants,

it is well known that
$$\alpha + \beta = \gamma + \delta \qquad (1)$$

Now suppose that the groundforms are of deg. order $(r.s)$, $(r'.s')$, . . . and let $(1 - a^r x^s)(1 - a^{r'} x^{s'})$ . . . be denoted by $\Pi(1 - a^r x^s)$, then
$$\frac{1}{\Pi(1 - a^r x^s)} = \Sigma(\gamma + \delta)\, a^m x^n \ldots \qquad (2)$$
In this we can by means of (1) replace $\gamma + \delta$ by $\alpha + \beta$, and if, moreover, we

---

* See the preceding note.

† To make my meaning perfectly clear, such a compound syzygant as $(AL + CH - DG)^2$, should it ever occur, is not considered except as a linear function of $AL(AL + CH - DG)$, $CH(AL + CH - DG)$, and $DG(AL + CH - DG)$.

‡ In this paper the word *syzygant* has the meaning originally given to it by Prof. Sylvester, *loc. cit.*

write $\Sigma \alpha a^m x^n = \phi(a, x)$, $\phi(a, x)$ is the generating function for covariants, and (2) becomes

$$\Sigma \beta a^m x^n = \frac{1}{\Pi(1 - a^r x^s)} - \phi(a; x) \tag{3}$$

This is the generating function for syzygants; its use is limited, by the one assumption made in finding it, to those quantics for which we have a *complete* and *correct* list of groundforms. It matters not how such a list may be found, but the proof of its correctness and completeness must be independent of the now disproved Fundamental Postulate of Tamisage.

4. The principle that the only compound syzygants that need be considered are linear functions of the simply divisible syzygants is of vital importance to the theory, and gives an immediate interpretation to each term in the numerator of the Generating Function for Syzygants. For if Σ denote a ground-syzygant, and A, B, C, . . . be the groundforms; all the syzygants that can be formed from Σ, *i. e.* all the simply divisible syzygants, are found in the expansion of $\frac{\Sigma}{(1-A)(1-B)(1-C)\ldots}$. Now the Generating Function for Syzygants can be resolved into a series of fractions of precisely this form, and hence the terms of the numerator will correspond to ground-syzygants.

It must be understood that there are syzygants not only of the first and second grades, but also, as Prof. Sylvester has remarked, of the 3d, 4th, and higher grades.

A Syzygant of the second grade is defined as a *linear* function of the simply divisible syzygants of the first grade, which is identically zero when considered as a function of the groundforms.

A Syzygant of the third grade is defined as a *linear* function of the simply divisible syzygants of the second grade, which is identically zero when considered as a function of the simply divisible syzygants of the first grade.

And generally—a Syzygant of the grade $n + 2$ is a *linear* function of the simply divisible syzygants of the grade $n + 1$, which is identically zero when considered as a function of the simply divisible syzygants of the grade n.

The Σ of the present article may be taken to be a syzygant of any grade, the first article of this paper contains examples of syzygants of the first grade, the second contains examples of syzygants of the second grade, and a simple example of a syzygant of the third grade occurs for the deg. order (11.24) for the binary sextic.

The sextic has in fact a syzygant of the second grade of deg. order (10.18), viz.

$$(10.18) \qquad \begin{aligned} & \mathrm{C}\,(\mathrm{AEK} - \mathrm{BEH} + \mathrm{CEG} + 6\mathrm{DQ} - 18\mathrm{FO}) \\ & - 6\mathrm{D}\,(\mathrm{CQ} + \mathrm{EO} - 3\mathrm{FN}) \\ & - \mathrm{E}\,(\mathrm{ACK} - \mathrm{BCH} + \mathrm{C}^2\mathrm{G} - 6\mathrm{DO} - 18\mathrm{FL}) \\ & + 18\mathrm{F}\,(\mathrm{CO} - \mathrm{DN} - \mathrm{EL}) = 0. \end{aligned}$$

In this, let the syzygants of the first grade, within brackets, be denoted by single letters, so that the syzygant of the second grade is

$$(10.18) \qquad \mathrm{C}\alpha - 6\mathrm{D}\beta - \mathrm{E}\gamma + 18\mathrm{F}\delta = \Gamma \text{ suppose.}$$

Treating the syzygants of the second grade, given in Art. 2, in a similar manner; the syzygants of deg. order (8.18), (8.22) and the first syzygants of deg. order (9.16) and (9.20) respectively give

$$\begin{aligned} &(8.18) \qquad \Delta = \mathrm{A}\delta - \mathrm{C}\varepsilon + \mathrm{D}\xi + \mathrm{E}\eta \\ &(8.22) \qquad \Theta = \mathrm{A}\gamma - \mathrm{C}\theta + 6\mathrm{D}\iota + 18\mathrm{F}\eta \\ &(9.16) \qquad \Lambda = \mathrm{A}\beta - \mathrm{C}\varkappa - \mathrm{E}\iota + 3\mathrm{F}\xi \\ &(9.20) \qquad \Omega = \mathrm{A}\alpha - 6\mathrm{D}\varkappa - \mathrm{E}\theta + 18\mathrm{F}\varepsilon. \end{aligned}$$

And then the Syzygant of the third grade referred to is

$$(11.24) \qquad \begin{aligned} & \mathrm{A}\Gamma - \mathrm{C}\Omega + 6\mathrm{D}\Lambda + \mathrm{E}\Theta - 18\mathrm{F}\Delta \\ = {} & \mathrm{A}\,(\mathrm{C}\alpha - 6\mathrm{D}\beta - \mathrm{E}\gamma + 18\mathrm{F}\delta) \\ - {} & \mathrm{C}\,(\mathrm{A}\alpha - 6\mathrm{D}\varkappa - \mathrm{E}\theta + 18\mathrm{F}\varepsilon) \\ + {} & 6\mathrm{D}\,(\mathrm{A}\beta - \mathrm{C}\varkappa - \mathrm{E}\iota + 3\mathrm{F}\xi) \\ + {} & \mathrm{E}\,(\mathrm{A}\gamma - \mathrm{C}\theta + 6\mathrm{D}\iota + 18\mathrm{F}\eta) \\ - {} & 18\mathrm{F}\,(\mathrm{A}\delta - \mathrm{C}\varepsilon + \mathrm{D}\xi + \mathrm{E}\eta) = 0. \end{aligned}$$

5. For the Binary Sextic, we have, see Prof. Sylvester's Tables of Generating Functions, etc. (*American Journal of Mathematics*, Vol. II, p. 225),

$$\Pi\,(1 - a^r x^s) = (1 - ax^6) \ldots (1 - a^{15}),$$

where there are 26 factors, one for each groundform. And $\phi\,(a, x) = \dfrac{N}{D}$, where

$$\begin{aligned} N = {} & 1 + a^3(x^2 + x^6 + x^8 + x^{12}) + a^4(x^4 + x^6 + x^{10}) + a^5(x^2 + x^4 + x^8 - x^{16}) \\ & + a^6(x^4 + 2x^6) + a^7(x^2 + x^4 + x^8 - x^{12}) + a^8(x^2 + x^4 + x^6 - x^{14}) \\ & + a^9(x^4 + x^6 - x^{10} - x^{12}) + a^{10}(x^2 + x^4 - x^{12} - x^{14}) + a^{11}(x^4 + x^6 - x^{10} - x^{12}) \\ & + a^{12}(x^2 - x^{10} - x^{12} - x^{14}) + a^{13}(x^4 - x^8 - x^{12} - x^{14}) + a^{14}(-2x^{10} - x^{12}) \\ & + a^{15}(1 - x^8 - x^{12} - x^{14}) + a^{16}(-x^6 - x^{10} - x^{12}) + a^{17}(-x^4 - x^8 - x^{10} - x^{14}) - a^{20}x^{16}. \end{aligned}$$

$$D = (1 - a^2)(1 - a^4)(1 - a^6)(1 - a^{10})(1 - a^2x^4)(1 - a^2x^8)(1 - ax^6).$$

Hence the numerator of the Generating Function for Syzygants $\frac{1}{\Pi(1-a^rx^s)}$ $-\phi(a, x)$, is

$$1-N\left\{\begin{array}{l}(1-a^3x^2)(1-a^3x^6)(1-a^3x^8)(1-a^3x^{12})(1-a^4x^4)(1-a^4x^6)(1-a^4x^{10})\\(1-a^5x^2)(1-a^5x^4)(1-a^5x^8)(1-a^6x^6)^2(1-a^7x^2)(1-a^7x^4)(1-a^8x^2)\\(1-a^9x^4)(1-a^{10}x^2)(1-a^{12}x^2)(1-a^{15})\end{array}\right\}$$

viz. N is multiplied by those factors of $\Pi(1-a^rx^s)$ which do not occur in D and the result subtracted from unity.

The result would be an expression of the 139th degree in a, and of the 106th in x, for the numerator. The denominator is of course $\Pi(1-a^rx^s)$.

The numerator for Syzygants of the Binary Sextic is

$$\begin{array}{l}a^5x^{16}+a^6(x^8+x^{10}+x^{12}+2x^{14}+x^{16}+x^{18}+x^{20}+x^{24})\\+a^7(x^6+x^{10}+4x^{12}+x^{14}+2x^{16}+2x^{18}+x^{22})\\+a^8(2x^8+4x^{10}+2x^{12}+4x^{14}+3x^{16}-x^{18}+2x^{20}-x^{22}-x^{24}-x^{28})\\+a^9(x^6+4x^8+2x^{10}+5x^{12}+3x^{14}-2x^{16}+2x^{18}-4x^{20}-4x^{22}-x^{24}-4x^{26}-x^{28}-x^{30}-x^{32})\\+\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\end{array}$$

The numerator of the Generating Function for Syzygants may be called the Numerator for Syzygants, for two reasons; first, for the sake of brevity, and secondly, since it gives the number of ground-syzygants of any grade. Results obtained from it are, however, liable to correction, whenever ground-syzygants of grade n coexist with ground-syzygants of grade $n+1$ of the same deg. order. Two such cases occur in the present example, viz. there is one ground-syzygant of the second grade and deg. order (9.14) and one of deg. order (9.18), so that the corresponding terms in the Numerator for Syzygants of the Sextic should be written $(4-1)a^9x^{14}+(3-1)a^9x^{18}$. When this is done the positive terms agree with the numbers given in Prof. Sylvester's Table of Syzygies (*American Journal of Mathematics*, Vol. IV, p. 59), except a^9x^6, where it will be shown hereafter that there is an error in the table; and the negative terms agree with the syzygants of the second grade given in Art. 2 of the present paper.

6. For Quantics which have groundforms and syzygies of the same deg. order a complete list of groundforms cannot be obtained by tamisage, and if $\Pi(1-a^rx^s)$ is formed from an incomplete list this will involve a correction of the Numerator for Syzygants.

In fact the preceding article shows that the Numerator for Syzygants is of the form $1-NM$, where N is the numerator of the generating function for covariants, in its representative form, and the multiplier M consists of those factors of $\Pi(1-a^r x^s)$ which do not appear in the denominator of the generating function for covariants. If then a factor $1-a^\rho x^\sigma$ has been omitted in $\Pi(1-a^r x^s)$ it will also have been omitted in M, and the corrected Numerator for Syzygants will be $1-NM(1-a^\rho x^\sigma)$. If more than one groundform has been omitted the correction will be of the same nature.

The correction for those cases in which Ground-Syzygants of grade n exist simultaneously with Ground-Syzygants of grade $n+1$, of the same deg. order, consists in multiplying N by each factor of M successively and separating those terms, as they occur, which in $1-NM$ will correspond to Syzygants of different grades, and thus preventing their combination with each other.

In the case of the Sextic the work may be arranged as follows:

$N = 1 +$

	+	+	+	+	−	−	−	−
a^3	x^2	x^6	x^8	x^{12}				
a^4	x^4	x^6	x^{10}					
a^5	x^2	x^4	x^8		x^{16}			
a^6	x^4	$2x^6$						
a^7	x^2	x^4	x^8		x^{12}			
a^8	x^2	x^4	x^6		x^{14}			
a^9	x^4	x^6			x^{10}	x^{12}		
a^{10}	x^2	x^4			x^{12}	x^{14}		
a^{11}	x^4	x^6			x^{10}	x^{12}		
a^{12}	x^2				x^{10}	x^{12}	x^{14}	
a^{13}	x^4				x^8	x^{12}	x^{14}	
a^{14}					$2x^{10}$	x^{12}		
a^{15}	1				x^8	x^{12}	x^{14}	
a^{16}					x^6	x^{10}	x^{12}	
a^{17}					x^4	x^8	x^{10}	x^{14}
a^{20}					x^{16}			

$N(1-a^3x^2) = 1 +$

	+	+	+	−	−	−	−	−	+	+	+	+
a^3	x^6	x^8	x^{12}									
a^4	x^4	x^6	x^{10}									
a^5	x^2	x^4	x^8	x^{16}								
a^6	$2x^6$			x^8	x^{10}	x^{14}						
a^7	x^2	x^4		x^6	$2x^{12}$							
a^8	x^2			x^{10}	x^{14}				x^{18}			
a^9	x^4			$2x^8$	x^{10}	x^{12}						
a^{10}	x^2			x^6	x^{10}	x^{12}	x^{14}		x^{14}			
a^{11}				x^8	x^{10}	x^{12}			x^{16}			
a^{12}	x^2			x^6	x^8	x^{10}	x^{12}	x^{14}	x^{12}	x^{14}		
a^{13}				x^6	x^8	x^{12}	x^{14}		x^{14}	x^{16}		
a^{14}				x^6	x^8	$2x^{10}$	x^{12}		x^{12}	x^{14}		
a^{15}	1			x^4	x^8	x^{12}	x^{14}		x^{12}	x^{14}	x^{16}	
a^{16}				$2x^6$	x^{10}	x^{12}			x^{10}	x^{14}	x^{16}	
a^{17}				x^4	x^8	x^{10}	x^{14}		$2x^{12}$	x^{14}		
a^{18}				x^2					x^{10}	x^{14}	x^{16}	
a^{19}									x^8	x^{12}	x^{14}	
a^{20}				x^{16}					x^6	x^{10}	x^{12}	x^{16}
a^{23}									x^{18}			

and so on until N has been multiplied by all the factors of M. In performing this first multiplication it has been assumed that all the negative terms of N correspond to ground-syzygants, and if this is not so, considerable uncertainty arises; thus if the term $-a^{10}x^{14}$ in N corresponds to a compound syzygant, formed by multiplying a ground-syzygant of deg. order (7.12) by the groundform of deg. order (3.2), it ought to be removed and the two terms $-a^{10}x^{14}$ and $+a^{10}x^{14}$ in $N(1-a^3x^2)$ ought to cancel one another; but if it corresponds to a ground-syzygant these two terms will correspond to a ground-syzygant of the first grade and one of the second grade respectively, and ought to be kept distinct. The same reasoning applies to the deg. orders (12.12), (12.14), (13.14), (14.12), (15.12), (15.14), (16.10), (17.14), and (20.16). A similar assumption is made at each successive multiplication, with the object of making the list of ground-syzygants of the first grade coincide, as closely as possible, with Prof. Sylvester's table, referred to above; and by these means the cases of disagreement have been reduced to three only; viz. there are found, one syzygy of deg. order (9.6) and one of deg. order (11.6), not given in the table, and two of deg. order (15.6) where the table gives three.

The factor $(1-a^3x^2)$ has been chosen for the first multiplication, because all those terms in the positive portion of N which correspond to compound forms can, by its use, be removed at one operation; and no other factor possesses this property. For the Quintic, the only factor possessing this property is $(1-a^5x)$; a circumstance of which Prof. Cayley has taken advantage in the arrangement of the terms in the numerator of his "Real Generating Function" (see his Tenth Memoir on Quantics, Phil. Trans. Part II, 1878, p. 608). In Quantics higher than the Sextic no such factor occurs.

The second, and all the remaining multiplications, are performed in the following manner.

Let $N(1-a^3x^2) = 1+\Gamma-\Sigma_1+\Sigma_2,$

then $N(1-a^3x^2)(1-a^rx^s) = 1+\Gamma'-\Sigma_1'+\Sigma_2'-\Sigma_3,$

where

$$\Gamma' = \Gamma - a^rx^s$$
$$\Sigma_1' = \Sigma_1 + a^rx^s\Gamma$$
$$\Sigma_2' = \Sigma_2 + a^rx^s\Sigma_1$$
$$\Sigma_3 = a^rx^s\Sigma_2.$$

Here Γ denotes the first positive block, whose terms consist exclusively of ground-types, Σ_1 denotes the negative block and typifies syzygants of the first grade (supposed to be ground-syzygants), and Σ_2 denotes the second positive block and is exclusively typical of syzygants of the second grade, provided those of Σ_1 are *ground*-syzygants, but not otherwise.

7. After the first multiplication, the method of the preceding article cannot be followed with certainty without a preliminary investigation to show precisely what the terms of each block denote (*e. g.* whether *all* those of the second block denote *ground*-syzygants), and hence what terms, if any, will cancel one another. But, in the case of the Sextic, since all the terms of the first block have been, by the first multiplication, made to typify groundforms only; all the terms which, originating in it, are placed in the second block after each multiplication, necessarily correspond to ground-syzygants. For they typify syzygants containing a term which is either the square of one or the product of two groundforms, and these cannot possibly be compound syzygants. Precisely similar reasoning shows that, after any block has been made to contain only terms typical of ground-syzygants of its own grade, all the terms which, originating in it at each multiplication, are transferred to the next block will necessarily typify ground-syzygants of the grade proper to the block to which they have been transferred. In fact no syzygant of grade $n+1$ can possibly be compound which contains, as one of its terms, a compound syzygant of grade n formed by the product of a ground-syzygant of grade n with a single groundform.

The form of the Generating Function, when corrected for cases of coexistence of a groundform and syzygant of the same deg. order, will perhaps place the whole in a clearer light.

Suppose then that, after all the groundforms that can be found by tamisage have been removed into the denominator, the Generating Function has been put in the form

$$\frac{1-\Sigma_1+\Sigma_2-\Sigma_3+\ldots}{\Pi(1-a^r x^s)},$$

the correction for the coexistence of a groundform and syzygant of deg. order $(\rho.\sigma)$ gives it the form

$$\frac{1-(a^\rho x^\sigma+\Sigma_1)+(a^\rho x^\sigma\Sigma_1+\Sigma_2)-(a^\rho x^\sigma\Sigma_2+\Sigma_3)+\ldots}{(1-a^\rho x^\sigma)\,\Pi(1-a^r x^s)}.$$

Now if Σ_1 contains one term which typifies a ground-syzygant (S) and another term which typifies a compound syzygant formed by multiplying S by the groundform $(\rho.\sigma)$, a term of the first block will cancel with one of the second, and this term of $a^\rho x^\sigma \Sigma_1$ has been wrongly placed in the second block; but if Σ_1 contains no such terms, every term of $a^\rho x^\sigma \Sigma_1$ has been rightly placed in the second block. An examination of the syzygants denoted by Σ_1 is therefore necessary to find out which of them, if any, is divisible by the groundform $(\rho.\sigma)$. The syzygants denoted by $a^\rho x^\sigma \Sigma_1$ may all of them be considered as compounds divisible by the groundform $(\rho.\sigma)$; but with this difference, those of them that are used to destroy compound syzygants of the block from which they originate, are considered as compound syzygants simply, but those that are transferred to the next block are considered *as single terms of syzygants of the next higher grade.* Hence when Σ_1 contains only ground-syzygants the compounds $a^\rho x^\sigma \Sigma_1$ will be the products of ground-syzygants with a single groundform, and having been all of them transferred to the next block, will stand (as sample terms) for *ground*-syzygants of the next higher grade. The reasoning is of course unaltered if we write Σ_n for Σ_1 throughout.*

I will conclude this section (which is an improvement on what I called the "Automatic Method of Tamisage" in the J. H. U. Circular for April, 1883), by remarking that $1 - \text{NM}$ (Art. 6) has no factor in common with $\Pi(1 - a^r x^s)$, and consequently the generating function for Syzygants is a fraction *in its lowest terms* whose denominator consists of factors typifying all the groundforms; but its very complicated form for the higher quantics sufficiently indicates the impracticability of finding this generating function by any method which does not involve a previous knowledge of the groundforms.

(III). *Exemplifications and Applications.*

8. It seems proper, at the commencement of this section, to give the leading terms of the covariants J′ and R′, and to state why they have been used instead of J and R.

Copying the leading term of J from Prof. Cayley's tables, and forming that of BC by multiplication, we have

---

* It was by such reasoning as this that I was led to consider the syzygant of the third grade of deg. order (11.24), given in Art. 4, a sample term of which is the ground-syzygant of deg. order (8.18) multiplied by the single groundform F.

	J, x^4	BC, x^4	J′, x^4
a^2ge		$+1$	$+1$
f^2	$+1$		-1
$abdg$		-4	-4
bef	-10	-6	$+4$
c^2g		$+3$	$+3$
cdf	$+4$		-4
ce^2	$+16$	$+15$	-1
d^2e	-12	-10	$+2$
a^0b^2df	$+16$	$+24$	$+8$
b^2e^2	$+9$		-9
bc^2f	-12	-18	-6
$bcde$	-76	-60	$+16$
bd^3	$+48$	$+40$	-8
c^3e	$+48$	$+45$	-3
c^2d^2	-32	-30	$+2$

Here $J' = BC - J$, and it is to be noticed that the part independent of a, or residue, of the source of O, is simply the residue of J′ multiplied by b. Such an arrangement, when possible, greatly facilitates the use of the method of residues. Thus (see Prof. Cayley's Tables for the Binary Sextic, *American Journal of Mathematics*, Vol. IV, p. 381 and 382),

$$\text{Res. G} = -\,2b.\text{Res. C}, \quad \text{and Res. O} = b.\text{Res. J}',$$

so that the syzygy $2CO + GJ' = AR'$ is immediately verified. An inspection of the syzygies given in Art. 1 of the present paper will show that this is not the only syzygy whose form has been simplified by the use of J′ and R′.

The source of R′ is given in the following table, together with Prof. Cayley's Q, R, and BK.

	Q, x^6	R, x^6	BK, x^6	R′, x^6
a^3dg^2		−1	+1	
efg		+9	−1	+1
f^3		−8		−1
a^2bcg^2		+3	−3	
$bdfg$		−24	−8	−4
be^2g		−45	+5	−5
bef^2		+66	+6	+9
c^2fg	−2	+3	+9	+3
$cdeg$	+5	+48	−2	+2
cdf^2	+6	−12		−6
ce^2f	−7	−51	−15	−3
d^3g	−3	−16	−2	
d^2ef	−3	+36	+10	+8
de^3	+4	−8		−4
ab^3g^2		−2	+2	
b^2cfg	+4	+12	+12	
b^2deg	−5	+192	+2	+28
b^2df^2	−6	−48	+12	
b^2e^2f	+7	−144	−30	−27
bc^2eg	−5	−159	−39	−21
bc^2f^2	−6	+18	−54	
bcd^2g	+7	−48	+26	−8
$bcdef$	−16	+24	+72	+24
bce^3	+23	+279	+75	+27
bd^3f	+30	−48	−28	−32
bd^2e^2	−33	−84	−50	+8
c^3dg	−1	+42		+6
c^3ef	+36	+153	+135	+9
c^2d^2f	−37	−36	−90	+12
c^2de^2	−53	−399	−255	−42
cd^3e	+79	+312	+290	+16
d^5	−24	−64	−80	
a^0b^4fg	−2		−12	
b^3ceg	+5		+30	
b^3cf^2	+6		+36	
b^3d^2g	+2	−224	−20	−32
b^3def	+22	+144	−12	
b^3e^3	−27	+54		+27
b^2c^2dg	−8	+336		+48
b^2c^2ef	−39	−108	−126	
b^2cd^2f	−50	+384	+84	+96
b^2cde^2	+107	−684	+30	−162
b^2d^3e	−22	+144	−20	+32
bc^4g	+3	−126		−18
bc^3df	+84	−648		−144
bc^3e^2	−21	+432	+90	+81
bc^2d^2e	−102	+564	−120	+132
bcd^4	+44	−288	+40	−64
c^5f	−27	+270		+54
c^4de	+45	−450		−90
c^3d^3	−20	+200		+40

It will be seen that $8R' = R + BK - 6Q$. The only apparent advantage in using R' is that it is somewhat simpler than R. The source of J' has also a special property which that of J does not possess, viz. operating on the invariant I, of the fourth degree, we have

$$(a\delta_c + 3b\delta_d + 6c\delta_e + 10d\delta_f + 15e\delta_g)\,I = \text{source of } J'.$$

9. In what follows a Syzygant will be denoted by one of its terms placed within $\overline{\underline{\quad}}$; thus the four Syzygants, for the Sextic, of deg. order (7.12) may be denoted by $\overline{\underline{AQ}}$, $\overline{\underline{AR'}}$, $\overline{\underline{DN}}$, and $\overline{\underline{HI}}$; but remembering what has been said above (Art. 7), this set of four syzygants is here represented by $\overline{\underline{DN}}$, $\overline{\underline{EL}}$, $\overline{\underline{FK}}$, and $\overline{\underline{GJ'}}$; and then $\overline{\underline{DN}}$ is the syzygant typified by the term $-a^7x^{12}$ in the unmultiplied numerator, and the pair of syzygants corresponding to $-2a^7x^{12}$ in the numerator after the first multiplication is $\overline{\underline{DN}}$, $\overline{\underline{EL}}$. This notation is followed, in the list of ground-syzygants given below, whenever the theory indicates a syzygant containing, as one of its terms, a binary combination of the groundforms of the numerator. When this is not the case the presence of syzygants is, in general, indicated by an asterisk; but it is not certain that all of these are *ground*-syzygants (see Art. 6). The five syzygants $\overline{\underline{DG}}$, $\overline{\underline{DN}}$, $\overline{\underline{DQ}}$, $\overline{\underline{DS}}$ and $\overline{\underline{DT}}$ have been inserted to complete the list as far as degree 9.

(5.16)	(6.8)	(6.10)	(6.12)	(6.14)	(6.16)	(6.18)	(6.20)	(6.24)
$\overline{\underline{DG}}$	$\overline{\underline{EF}}$	$\overline{\underline{EG}}$	$\overline{\underline{F^2}}$	$\overline{\underline{EH}}$	$\overline{\underline{G^2}}$	$\overline{\underline{FH}}$	$\overline{\underline{GH}}$	$\overline{\underline{H^2}}$
				$\overline{\underline{FG}}$				

(7.6)	(7.10)	(7.12)	(7.14)	(7.16)	(7.18)	(7.22)	(8.8)	(8.10)	(8.12)	(8.14)	(8.16)	(8.20)
$\overline{\underline{EJ'}}$	$\overline{\underline{FJ'}}$	$\overline{\underline{DN}}$	$\overline{\underline{GK}}$	$\overline{\underline{FL}}$	$\overline{\underline{GL}}$	$\overline{\underline{HL}}$	$\overline{\underline{FM}}$	$\overline{\underline{EO}}$	$\overline{\underline{GN}}$	$\overline{\underline{DQ}}$	$\overline{\underline{GO}}$	$\overline{\underline{HO}}$
		$\overline{\underline{EL}}$		$\overline{\underline{HJ'}}$	$\overline{\underline{HK}}$		$\overline{\underline{J'^2}}$	$\overline{\underline{FN}}$	$\overline{\underline{K^2}}$	$\overline{\underline{FO}}$	$\overline{\underline{HN}}$	$\overline{\underline{L^2}}$
		$\overline{\underline{FK}}$						$\overline{\underline{GM}}$		$\overline{\underline{HM}}$	$\overline{\underline{KL}}$	
		$\overline{\underline{GJ'}}$						$\overline{\underline{J'K}}$		$\overline{\underline{J'L}}$		

(9.6)	(9.8)	(9.10)	(9.12)	(9.14)	(9.18)	(10.4)	(10.6)	(10.8)	(10.10)	(10.12)	(10.14)	(10.16)
$\overline{\underline{J'M}}$	$\overline{\underline{EQ}}$	$\overline{\underline{DS}}$	$\overline{\underline{DT}}$	$\overline{\underline{GQ}}$	$\overline{\underline{HQ}}$	$\overline{\underline{M^2}}$	$\overline{\underline{ET}}$	$\overline{\underline{FS}}$	*	*	*	$\overline{\underline{HT}}$
	$\overline{\underline{ER'}}$	$\overline{\underline{KN}}$	$\overline{\underline{FQ}}$	$\overline{\underline{GR'}}$	$\overline{\underline{HR'}}$		$\overline{\underline{MN}}$	$\overline{\underline{N^2}}$	$\overline{\underline{FT}}$	$\overline{\underline{GT}}$	$\overline{\underline{HS}}$	$\overline{\underline{LQ}}$
	$\overline{\underline{J'N}}$		$\overline{\underline{FR'}}$	$\overline{\underline{KO}}$	$\overline{\underline{LO}}$				$\overline{\underline{GS}}$	$\overline{\underline{KQ}}$		$\overline{\underline{LR'}}$
	$\overline{\underline{KM}}$		$\overline{\underline{J'O}}$	$\overline{\underline{LN}}$					$\overline{\underline{J'Q}}$	$\overline{\underline{KR'}}$		$\overline{\underline{O^2}}$
			$\overline{\underline{LM}}$						$\overline{\underline{J'R'}}$	$\overline{\underline{NO}}$		
									$\overline{\underline{MO}}$			

(11.6)	(11.8)	(11.10)	(11.12)	(11.14)	(12.4)	(12.6)	(12.8)	(12.10)	(12.12)	(12.14)	(12.16)
J'S	*	*	*	HU	MS	EV	*	*	*	*	HV
	FU	GU	LS	LT		J'U	KU	FV	GV		
	J'T	KT		OQ		MT	NT	OS	LU		
	KS	NQ		OR'		NS			OT		
	MQ	NR'							Q^2		
	MR'								QR'		
									R'^2		

(13.4)	(13.6)	(13.8)	(13.10)	(13.12)	(13.14)	(14.4)	(14.6)	(14.8)	(14.10)	(14.12)
MU	*	*	GX	*	*	S^2	*	*	*	*
	NU	FX	KV		HX		J'X	KX	*	LX
		J'V	OU		LV		MV	NV		OV
		QS	QT				ST	QU		
		R'S	R'T					R'U		
								T^2		

(15.4)	(15.6)	(15.8)	(15.10)	(15.12)	(15.14)	(16.4)	(16.6)	(16.8)	(16.10)	(16.12)
EY	NX	*	GY	*	*	U^2	*	KY	*	*
MX	TU	FY	OX		HY		*	QX		LY
SU			QV				J'Y	R'X		
			R'V				SV	TV		

(17.4)	(17.6)	(17.8)	(17.10)	(17.14)	(18.2)	(18.4)	(18.6)	(18.8)	(18.12)
*	NY	*	*	*	EZ	UX	FZ	GZ	HZ
MY	TX		OY					QY	
SX	UV							R'Y	
								V^2	

(19.4)	(19.6)	(19.10)	(20.2)	(20.4)	(20.8)	(20.16)	(21.6)
J'Z	KZ	LZ	MZ	NZ	OZ	*	QZ
	TY			UY			R'Z
	VX			X^2			VY

(22.2)	(22.4)	(23.2)	(24.4)	(25.2)	(27.2)	(30.0)
SZ	TZ	UZ	VZ	XZ	YZ	Z^2
	XY		Y^2			

If we now assume that each asterisk corresponds to a *ground*-syzygant, which is in fact the same as the assumption made in Art. 6, the list gives exactly the same results as Prof. Sylvester's table, except in the three cases already noticed.

10. When certain fundamental syzygants are known, all the rest may be calculated very easily by common algebra. The method will be applied, in this article, to the calculation of the syzygants of degree 9, for the Sextic.

We have, from Art. 1, using the notation of Art. 9,

$$\begin{aligned}
J'\,\overline{\underline{EF}} &= J'(AM - 2ABE + B^2D - BC^2 - 3CJ' - 36DI + 36EF)\\
AB\,\overline{\underline{EJ'}} &= AB(9AP + 12ABI - 3B^2F + 4BCE - CM - 12EJ' + 108FI)\\
B^2\,\overline{\underline{F^2}} &= B^2(3A^2I - 3ABF + 2ACE + BCD - C^3 - 3DJ' + 27F^2)\\
BC\,\overline{\underline{EF}} &= BC(AM - 2ABE + B^2D - BC^2 - 3CJ' - 36DI + 36EF)\\
C\,\overline{\underline{J'^2}} &= C(AS + 4C^2I - FM - J'^2)\\
F\,\overline{\underline{EJ'}} &= F(9AP + 12ABI - 3B^2F + 4BCE - CM - 12EJ' + 108FI)\\
I\,\overline{\underline{F^2}} &= I(3A^2I - 3ABF + 2ACE + BCD - C^3 - 3DJ' + 27F^2)
\end{aligned}$$

Whence, $3J'\,\overline{\underline{EF}} - AB\,\overline{\underline{EJ'}} + B^2\,\overline{\underline{F^2}} - BC\,\overline{\underline{EF}} - 9C\,\overline{\underline{J'^2}} + 9F\,\overline{\underline{EJ'}} - 36I\,\overline{\underline{F^2}}$

$$= 3A(J'M - 3AB^2I - 3ABP - 36AI^2 + 2BEJ' + 36BFI - 24CEI - 3CS + 27FP)$$

Now, since the left-hand side of this is a compound syzygant, the right side is one also; and dividing by 3A we have the value of the syzygant (9.6) or $\overline{\underline{J'M}}$, given in the list, but not in Prof. Sylvester's table. At the same time is found the ground-syzygy of the second grade

$$3J'\,\overline{\underline{EF}} - AB\,\overline{\underline{EJ'}} + B^2\,\overline{\underline{F^2}} - BC\,\overline{\underline{EF}} - 9C\,\overline{\underline{J'^2}} + 9F\,\overline{\underline{EJ'}} - 36I\,\overline{\underline{F^2}} = 3A\,\overline{\underline{J'M}}. \tag{10.12}$$

Again if

$$\begin{aligned}
C\,\overline{\underline{EO}} &= C(AT - 2EO - J'K)\\
C\,\overline{\underline{J'K}} &= C(2B^2L - BCK - 3BEG - 18EO + GM - 72IL + 3J'K)\\
E\,\overline{\underline{EL}} &= E(AQ + 2EL - 3FK)\\
E\,\overline{\underline{FK}} &= E(ABK - B^2H + BCG - 6CO - 12EL - 18FK + 36HI)\\
E\,\overline{\underline{GJ'}} &= E(AR' - GJ' - 2CO)\\
G\,\overline{\underline{EJ'}} &= G(9AP + 12ABI - 3B^2F + 4BCE - CM - 12EJ' + 108FI)\\
I\,\overline{\underline{EH}} &= I(AO - DK + EH)\\
I\,\overline{\underline{FG}} &= I(AO - 2CL + 3FG)\\
J'\,\overline{\underline{EG}} &= J'(AN - CK + EG)\\
K\,\overline{\underline{EF}} &= K(AM - 2ABE + B^2D - BC^2 - 3CJ' - 36DI + 36EF)\\
B^2\,\overline{\underline{EH}} &= B^2(AO - DK + EH)\\
B^2\,\overline{\underline{FG}} &= B^2(AO - 2CL + 3FG)\\
BC\,\overline{\underline{EG}} &= BC(AN - CK + EG)
\end{aligned}$$

we have $E\,\overline{\underline{GJ'}} + J'\,\overline{\underline{EG}} - C\,\overline{\underline{EO}} = A(ER' + J'N - CT)$

$$\begin{aligned}
&B^2\,\overline{\underline{FG}} - 36I\,\overline{\underline{FG}} - BC\,\overline{\underline{EG}} + 3C\,\overline{\underline{EO}} + C\,\overline{\underline{J'K}} + G\,\overline{\underline{EJ'}} - 12E\,\overline{\underline{GJ'}}\\
&\qquad = A(B^2O - 36IO - BCN + 3CT + 9GP + 12BGI - 12ER')\\
&B^2\,\overline{\underline{EH}} - 36I\,\overline{\underline{EH}} - BC\,\overline{\underline{EG}} - 3C\,\overline{\underline{EO}} + E\,\overline{\underline{FK}} + K\,\overline{\underline{EF}} + 6E\,\overline{\underline{EL}}\\
&\qquad = A(B^2O - 36IO - BCN - 3CT - EBK + KM + 6EQ).
\end{aligned}$$

Here the method only gives three of the four syzygies; the fourth is a fundamental one, introducing the covariant U for the first time, and must be found independently. It is

$$KM + 2E(BK - 3R' - 6Q) = 3AU.$$

There is no other fundamental syzygy of degree 9 but this; all the others are found by a mere repetition of the work given above, and are therefore non-fundamental; to each of them corresponds a ground-syzygant of the second grade of degree 10.

The complete list of syzygies of degree 9 is here given, the corresponding syzygies of the second grade are omitted for the sake of brevity. The names correspond to the list of Art. 9.

Deg. Order.	Name.	
(9.6)	$\overline{\underline{J'M}}$	$3AB^2I+3ABP+36AI^2-2BEJ'-36BFI+24CEI+3CS-27FP-J'M=0$
(9.8)	$\overline{\underline{EQ}}$	$B^2O-36IO-BCN-3CT-BEK+KM+6EQ=0$
(9.8)	$\overline{\underline{ER'}}$	$B^2O-36IO-BCN+3CT+9GP+12BGI-12ER'=0$
(9.8)	$\overline{\underline{J'N}}$	$CT-ER'-J'N=0$
(9.8)	$\overline{\underline{KM}}$	$3AU-2BEK+6ER'+12EQ-KM=0$
(9.10)	$\overline{\underline{DS}}$	$ABCI+ACP-AIJ'+BFJ'-2CEJ'-E^2F+DS=0$
(9.10)	$\overline{\underline{KN}}$	$12ABCI+ABE^2+9ACP-AEM-6CEJ'-18E^2F+3KN=0$
(9.12)	$\overline{\underline{DT}}$	$BEL-2CEK-E^2G-3J'O+3DT=0$
(9.12)	$\overline{\underline{FQ}}$	$3AIK+BCO-3BFK-C^2N+3CEK-3DT+9FQ=0$
(9.12)	$\overline{\underline{FR'}}$	$2BCO-2C^2N+6CGI-3J'O-9FR'=0$
(9.12)	$\overline{\underline{J'O}}$	$6AIK-2BEL-3BFK-12BHI+7CEK+2E^2G-9HP-6J'O=0$
(9.12)	$\overline{\underline{LM}}$	$AEN-6CGI-3J'O+LM=0$
(9.14)	$\overline{\underline{GQ}}$	$2LN-KO-GQ=0$
(9.14)	$\overline{\underline{GR'}}$	$ABCJ'-8AC^2I-AJ'^2+2BC^2F-4C^3E-6CFJ'-GR'=0$
(9.14)	$\overline{\underline{KO}}$	$2A^2EI-4ABDI-ABEF+ACE^2-3ADP+2DEJ'-KO=0$
(9.14)	$\overline{\underline{LN}}$	$A^2EI-ABEF-2AC^2I+3CFJ'+9EF^2-LN=0$
(9.18)	$\overline{\underline{HQ}}$	$AK^2-BHK+CGK-12LO+6HQ=0$
(9.18)	$\overline{\underline{HR'}}$	$BGL-2CGK-EG^2+6LO+3HR'=0$
(9.18)	$\overline{\underline{LO}}$	$2ACDI+ACEF-3DFJ'+LO=0$

11. The arguments in favor of the completeness of the list of formulæ given in this paper may be summarised as follows. The "Numerator for Syzygants," (Art. 5) gives an inferior limit to their number; the corrections of Art. 6 give, on the other hand, a superior limit, and actual calculation shows that this superior limit is attained in every case hitherto considered. It is, however, extremely doubtful whether this will happen for degress from 10 to 17 inclusive, or for the deg. order (20.16); for degrees above 20, and for 18 and 19, the superior and inferior limits coincide, as will be seen by an inspection of the asterisks of Art. 9.